A class of Fokker-Planck equations with logarithmic factors in diffusion and drift terms

2002 J. Phys. A: Math. Gen. 35 1827

(http://iopscience.iop.org/0305-4470/35/8/303)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **35** (2002) 1827–1832 PII: S0305-4470(02)31916-4

A class of Fokker–Planck equations with logarithmic factors in diffusion and drift terms

Karol Pesz

Institute of Physical and Theoretical Chemistry, Technical University of Wrocław, 50–370 Wrocław, Wybrzeże Wyspiańskiego 27, Poland

Received 6 December 2001
Published 15 February 2002
Online at stacks.iop.org/JPhysA/35/1827

Abstract

The $x^2 \ln(x)$ dependence of the diffusion coefficient in the Fokker–Planck equation is retrieved by means of symmetry arguments. Exact solutions of the equation with logarithmic factors in coefficients are presented. Algebraic and log-algebraic solutions are found. For some values of exponents they seem to be analogues (on an interval) of log-normal distributions.

PACS numbers: 05.10.Gg, 02.20.-a, 02.20.Jr. 02.20.Rt, 47.45.Dt, 66.10.C

Fokker–Planck equations (FPEs) with variable coefficients are widely investigated and apply in many situations. To name a few, let us recall the problem of diffusion in colloids [1], short-time chemical reactions [2], self-propelling particles [3], plasma physics [4], financial markets [5] and quantum chaos [6]. In each of the cases mentioned, if an FPE is invoked (and if nonlinearities are neglected) it contains a kind of space- (or space-like) and/or time-dependent diffusion coefficient (at least at some, usually short, timescales). Commonly, in various applications polynomial dependences of $D(x)$ are met.

The FPE with non-constant coefficients, usually written in the form [7–9]

$$u_t = \{D(x,t)u_x\}_x + (F(x,t)u)_x \tag{1}$$

adopts a more compact shape

$$\tilde{u}_{t'} = \tilde{u}_{yy} + (\tilde{F}(y,t')\tilde{u})_y \tag{2}$$

when time dependence of $D(x,t)$ is scaled out and the following change of variables is applied (see [10] and references therein):

$$u = \frac{\tilde{u}}{\sqrt{D'(x')}} \qquad y = \int \frac{\mathrm{d}x'}{\sqrt{D'(x')}}. \tag{3}$$

In equations (2) and (3) the primed variables are those obtained after a change of variables in the time domain (as an example, see equations (17) and (18)) and

$$\tilde{F}(y,t') = \frac{1}{2}[\ln D'(y)]_y + \frac{F'(y,t')}{\sqrt{D'(y)}}. \tag{4}$$

The issue of what sort of function of x and t may appear as diffusion and drift coefficients in (1) has been investigated many times. The most interesting point is, apart from their physical meaning, how they affect symmetry properties of the equation. Cicogna and Vitali [11] considered FPEs with symmetry group locally isomorphic to the group of the heat equation and provided the condition for six- and fourfold symmetries; they illustrated their approach with rather undemanding cases. Stehlen and Stogny [12] collected a few examples of FPEs customarily applied in science and Rudra [13] considered general rules for finding six- and four-dimensional groups for x-dependent coefficients, however without referring to any particular case. Also, potential symmetries of FPEs were discussed by Pucci and Saccomandi [14] but the authors discussed in detail a simple example.

Recently, Spichak and Stognii (SS) [10] summarized in an elegant way methods for symmetry properties of FPEs with variable coefficients and reported a few examples of equation (1) that can be reduced to the heat equation or to an equation with fourfold symmetry. The authors also allowed for a time dependence of the diffusion and drift terms. In effect, they considered in detail the time-dependent drift term. Here, we would like to point out that consideration of $D(x, t)$ provides alternative, more refined information. This gives the diffusion coefficient containing a logarithmic factor. Such a dependence, once spotted on the grounds of symmetry considerations, can be treated by the methods presented in [10]; the other way round seems, however, a quite intricate issue.

The most interesting cases of an FPE bearing the four ($\kappa \neq 0$) or six ($\kappa = 0$) infinitesimal generators result when $\tilde{F}(y, t)$ (equation (4)) satisfies the following condition:

$$\int \tilde{F}_t \, \mathrm{d}y - \tilde{F}_y - \frac{1}{2}\tilde{F}^2 = \frac{\kappa}{(y + a_3(t))^2} + a_2(t)y^2 + a_1(t)y + a_0(t) \tag{5}$$

as SS proved in [10] for a time-dependent case. For the static drift terms this has been stated several times before [11, 18].

We explore a condition similar to formula (5) but expressed in the original variable x and applied to the diffusion term, $D(x, t)$. Details and formulae *in extenso* will be given elsewhere. It turns out that other than polynomial dependences also admit high-symmetry cases. This conclusion is reached by considering point symmetries for the original equation (1) while taking advantage of results concerning potential symmetries for the constraints. In principle, the formulae for symmetry conditions of an FPE are equivalent in x and y variables. However, as they differ slightly in structure, they prompt us to perform distinct predictions about possible shapes of coefficients. For example, while the symmetry requirements as stated in terms of the drift coefficient, F, depend only on the first time derivative, similar conditions as expressed for D depend on the second time derivative.

Applying standard methods of classical Lie group symmetry analysis [21, 22] we found that for $F = 0$ a six-dimensional group of the free diffusion equation is admitted when the following equation is satisfied:

$$f_{tt} + \{f^{-2}(f^{-1})_{xx}\}_{xx} = \sigma^2(t)f \tag{6}$$

where $f(x, t) = 1/\sqrt{D(x, t)}$ and $\sigma(t)$ is an arbitrary function of time. Although equation (6) represents only a special case of relation (5) it provides interesting clues. In a fully developed form it reads

$$4\sigma^2(t)D^2 + 2DD_{tt} - 3D_t^2 = 2D^3D_{xxxx} - D^2D_{xx}^2 + 2DD_x^2D_{xx} - \tfrac{3}{4}D_x^4. \tag{7}$$

Equation (6) can be obtained as a time derivative of a simpler equation

$$\tilde{\sigma}(t)f = f_t \pm \left(\frac{1}{f}\right)_{xx} \tag{8}$$

where a new function $\tilde{\sigma}^2(t) = \sigma_t + \sigma^2(t)$. For a special case of $\tilde{\sigma} = 0$ the equation takes a particularly simple form

$$f_t = \mp(f^{-2} f_x)_x. \tag{9}$$

This is a specific diffusion equation belonging to the class of nonlinear reaction–diffusion equations. Solutions to this equation, expressed in an implicit way, were discussed by Hill [16] and King [17]. On the other hand, some of the solutions expressed by elementary functions were found by Gandarias [18] by means of potential symmetries. Similar solutions are also given by Kara *et al* [19]. The solutions are of the form

$$f^{-2} = \lambda(t)x^2 \ln[x\mu(t)]. \tag{10}$$

Equation (8) when written in terms of D

$$D_t = \pm D D_{xx} \mp \tfrac{1}{2} D_x^2 - 2\sigma(t) D \tag{11}$$

belongs to a class of equations known as porous medium equations and are known to possess few analytical solutions. Recently, Zhdanov and Andreitsev [20] considered this type of equation by means of higher conditional symmetries.

According to the guess (10), we substitute $D(x,t) = 4\lambda(t)x^2 \ln(x\mu(t))$ into (7) obtaining conditions for $\lambda(t)$ and $\mu(t)$

$$2\mu_t = \pm\lambda\mu \tag{12a}$$

$$\lambda_{tt} = \frac{1}{2}\lambda^3 + \frac{3}{2\lambda}\lambda_t^2 - 2\sigma^2\lambda. \tag{12b}$$

Let us note that all the examples cited in [10] are those with polynomial $D(x)$; in these cases solutions are akin to log-normal distributions. Here, we obtained the contribution with a logarithmic factor. An FPE with the drift term containing a factor depending logarithmically on x was also discussed by Lehnigk [15]

$$u_t = \{b_1 x^2 u_x + (b_2 x - b_3 x \ln x)u\}_x. \tag{13}$$

The equation can be transformed into a heat equation and the shape of the resulting probability density function is that for a log-normal distribution. Logarithmic factors are scarcely taken into account in FPEs. Here, it is demonstrated that a logarithmic term can also appear as a diffusion coefficient and solutions have the form of algebraic and log-algebraic distributions.

Before proceeding further let us consider the equation

$$u_t = -\{(4x^2 \ln x)u_x\}_x \tag{14}$$

where $0 \leqslant x \leqslant 1$ and the minus sign is taken to keep the diffusion coefficient positive. The change of variables (3) gives $y = \sqrt{-\ln x}$, $D(y) = 4y^2 e^{-2y^2}$ and equation (14) adopts the form of that describing the Rayleigh process

$$\tilde{u}_t = \tilde{u}_{yy} - \left[\left(\frac{1}{y} - 2y\right)\tilde{u}\right]_y \tag{15}$$

and as the condition (5) reads

$$\tilde{F}_y + \frac{1}{2}\tilde{F}^2 = -\frac{1}{2y^2} + 2y^2 - 4 \tag{16}$$

this indicates that the symmetry group is four-dimensional. On the other hand, the equation with a time-dependent D (without loss of generality $\lambda(t)$ can be set to unity in (10))

$$u_t = -4\{x^2 \ln[x\mu(t)]u_x\}_x \tag{17}$$

which after a change of variables $x' = x\mu(t)$, $t' = t$ takes the form

$$u_{t'} = -4\{x'^2 \ln x' u_{x'}\}_{x'} + \frac{\mu_{t'}}{\mu} x' u_{x'}. \tag{18}$$

This equation can be mapped into a heat equation for a particular value of $\mu_{t'}/\mu = 2$. This is an example of temporal dependence of D affecting symmetry of FPE. This is rather unusual as the time dependence (according to equation (5)) cannot change the dimension of the symmetry algebra. Here, it seems to be accidental and is related to the presence of the logarithmic factor.

Now, let us consider a balance between the logarithm-containing contributions in both diffusion and drift terms. When transformation (3) is applied to the equation

$$u_t = -\{(4x^2 \ln x)u_x\}_x + \{(Ax + Bx \ln x)u\}_x \tag{19}$$

the problem of finding solutions to this equation is equivalent to solving the following equation:

$$\tilde{u}_t = \tilde{u}_{yy} + \left\{ \left(\frac{\alpha}{y} + \beta y \right) \tilde{u} \right\}_y \tag{20}$$

where $\alpha = -(2 + A)/2$, $\beta = (B + 4)/2$.

Special cases appear for values of α equal to 2 and 0. The value $\alpha = 0$ reduces equation (20) to a regular Ornstein–Uhlenbeck case while $\alpha = 2$ relates to a Rayleigh-type process, an equation reducible to the heat equation.

For $\alpha \neq \{2, 0\}$ equation (20) is an extension of equations describing the Rayleigh or Rayleigh-like processes. Its stationary solution has the form of the Wigner–Dyson distribution [6]

$$\tilde{u}_s(y) = y^{-\alpha} \exp\left(-\frac{\beta}{2} y^2 \right). \tag{21}$$

In terms of the original variable, x, the distribution reads

$$u_s(x) = u_0 x^{B/4} (-\ln x)^{A/4}. \tag{22}$$

The shape of this function strongly depends on the values of parameters A and B. For both A and B negative the function has singularities at the ends of [0, 1] interval, whereas for positive values its shape resembles that for the log-normal distribution and can be normalized.

Having obtained an integrable form of FPE with the help of symmetry analysis we can safely look for some exact similarity solutions. The generators spanning the four-parameter Lie group of equation (20) are

$$\begin{aligned} &T = \partial_t \qquad U = u\partial_u \\ &V = \exp(2\beta t)\{T + \beta(\alpha - \beta y^2)U + \beta y \partial_y\} \\ &W = \exp(-2\beta t)\{T + \beta U - \beta y \partial_y\} \end{aligned} \tag{23}$$

and commutation relations for them read

$$\begin{aligned} &[V, W] = -4\beta T + \beta(2 - \alpha)U \\ &[V, T] = -2\beta V \qquad [W, T] = 2\beta W \\ &[V, U] = [W, U] = [T, U] = 0. \end{aligned} \tag{24}$$

Looking for solutions invariant under action of V for the symmetry variable $s = y \exp(-\beta t)$ we have to solve an equation

$$2s^2 Q_{ss} + \alpha s Q_s - \alpha Q = 0 \tag{25}$$

for the function $Q(s)$ satisfying

$$\tilde{u}(y, t) = Q(s) \exp(-\alpha\beta t - \beta y^2/2). \tag{26}$$

On using the general form of solution to equation (25)

$$Q(s) = C_1 s + C_2 s^{-\alpha} \tag{27}$$

we are left with the solution

$$\tilde{u}(y, t) = \{C_1 y \exp(-\beta(1 + \alpha)t) + C_2 y^{-\alpha}\} \exp(-\beta y^2/2) \tag{28}$$

which also contains the stationary solution (21).

On the other hand, as to solutions generated by linear combinations of generators we found an interesting solution belonging to the $V + W$ symmetry. Solving the system of characteristic equations we obtain for the similarity variable

$$s = \frac{y^2}{\cosh(2\beta t)} \tag{29}$$

a solution $\tilde{u}(y, t)$ in the form

$$\tilde{u}(y, t) = Q(s)[\cosh(2\beta t)]^{(1-\alpha)/4} \exp\left\{\frac{\beta}{2}\left((1-\alpha)t - \frac{y^2}{2}(1 + \tanh(2\beta t))\right)\right\} \tag{30}$$

with function $Q(s)$ satisfying

$$2s^2 Q_{ss} + s(1-\alpha)Q_s + \left(\frac{\alpha}{2} + \frac{\beta^2}{8}s^2\right)Q = 0. \tag{31}$$

General solutions of equation (31) are expressed in terms of linear combinations of Bessel functions of fractional order. Hence, for the similarity variable as given by (29) the solution of equation (20) reads

$$\begin{aligned} \tilde{u}(y, t) = y^{(1-\alpha)/2}[\cosh(2\beta t)]^{-1/2}&\left\{C_1 j_\nu\left(\frac{\beta s}{4}\right) + C_2 y_\nu\left(\frac{\beta s}{4}\right)\right\} \\ &\times \exp\left\{\frac{\beta(1-\alpha)t}{2} - \frac{\beta y^2}{4}[1 + \tanh(2\beta t)]\right\} \end{aligned} \tag{32}$$

where $j_\nu(z)$ and $y_\nu(z)$ are Bessel functions of non-integer order with the value of index $\nu = -(1 + \alpha)/4$; C_1, C_2 represent other constants of integration.

In summary, we have found a class of FPEs with a logarithmic inhomogeneity of the diffusion coefficient. Such an extension of a common quadratic dependence seems to be 'natural' as it is provided by symmetry arguments although on the physical side it may be difficult to justify it from a Langevin equation [7]. Curiously enough, such dependences have been spotted several times in various contexts. For example, Jeffrey and Onishi [23] found a $(1 - x)^2 \ln(1 - x)$ dependence of the mobility function for diffusing hard spheres. A similar shape of $D(n)$ appeared for quantum diffusion [24]. Also, closer inspection of the second moment as estimated from a conditional probability density function for increments in price changes [25] shows deviations from a quadratic dependence that may suggest a logarithmic component for small Δx.

Acknowledgments

I would like to acknowledge the hospitality of Brunel University, London, where my interest in this topic was aroused during a visit supported by the British Council and the Committee for Scientific Research within the British–Polish Joint Research Collaboration Programme. I am also grateful to Dr Barbara J Gabrys for her continuous support of the work.

References

[1] Segrè P N and Pusey P N 1996 Scaling of the dynamic scattering function of concentrated colloidal suspensions *Phys. Rev. Lett.* **77** 771–4

[2] Dong W and Andre J C 1994 Diffusion-controlled reactions. II. An approach based on a generalized diffusion equation *J. Chem. Phys.* **101** 299–306

[3] Li J H and Huang Z Q 1998 Transport of particles caused by correlation between additive and multiplicative noise *Phys. Rev.* E **57** 3917–22

[4] Navratil G A and Post R S 1997 Observation of classical diffusion in the presence of large amplitude density fluctuations *Phys. Lett.* A **64** 223–7

[5] Sornette D 2001 Fokker–Planck equation of distributions of financial returns and power laws *Physica* A **290** 211–17

[6] Chirikov B V 1992 The problem of quantum chaos *Chaos and Quantum Chaos (Lecture Notes in Physics)* ed W Dieter Heiss (Berlin: Springer) pp 1–56

[7] van Kampen N G 1987 *Stochastic Processes in Physics and Chemistry* (Amsterdam: Elsevier)

[8] Gardiner C 1985 *Handbook of Stochastic Methods* (Berlin: Springer)

[9] Risken H 1989 *The Fokker–Planck Equation: Methods of Solution and Applications (Springer Series in Synergetics vol 18)* (Berlin: Springer)

[10] Spichak S and Stognii V 1999 Symmetry clasification and exact solutions of the one-dimensional Fokker–Planck equation with arbitrary coefficients of drift and diffusion *J. Phys. A: Math. Gen.* **32** 8341–53

[11] Cicogna G and Vitali D 1989 Generalized symmetries of Fokker–Planck-type equations *J. Phys. A: Math. Gen.* **22** L453–6

Cicogna G and Vitali D 1990 Classification of the extended symmetries of Fokker–Planck equations *J. Phys. A: Math. Gen.* **23** L85–8

[12] Shtelen W M and Stogny V I 1989 Symmetry properties of one- and two-dimensional Fokker–Planck equations *J. Phys. A: Math. Gen.* **22** L539–43

[13] Rudra P 1990 Symmetry classes of Fokker–Planck-type equations *J. Phys. A: Math. Gen.* **23** 1663–70

[14] Pucci E and Saccomandi G 1993 Potential symmetries and solutions by reduction of partial differential equations *J. Phys. A: Math. Gen.* **23** 681–90

[15] Lehnigk S H 1989 On the log-normal diffusion process *J. Math. Phys.* **30** 953–5

[16] Hill J M 1989 Similarity solutions for nonlinear diffusion—a new integration procedure *J. Eng. Math.* **23** 141–55

Hill D L and Hill J M 1990 Similarity solutions for nonlinear diffusion—further exact results *J. Eng. Math.* **24** 109–24

[17] King J R 1990 Exact similarity solutions to some nonlinear diffusion equations *J. Phys. A: Math. Gen.* **23** 3681–97

[18] Gandarias M L 1996 Potential symmetries of a porous medium equation *J. Phys. A: Math. Gen.* **29** 5919–34

[19] Kara A H, Mahomed F M and Changzheng Qu 2000 Approximate potential symmetries for partial differential equations *J. Phys. A: Math. Gen.* **33** 6601–13

[20] Zhdanov R Z and Andreitsev A Yu 2000 Non-classical reduction of initial-value problems for a class of nonlinear evolution equations *J. Phys. A: Math. Gen.* **33** 5763–81

[21] Olver P J 1986 *Applications of Lie Groups to Differential Equations* (Berlin: Springer)

[22] Bluman G W and Cole J D 1974 Similarity methods for differential equations *Applied Mathematical Sciences* vol 13 (Berlin: Springer)

[23] Jeffrey D J and Onishi Y 1984 Calculation of resistance and mobility functions for two unequal rigid spheres in low-Reynolds-number flow *J. Fluid Mech.* **139** 261–90

[24] Dettman C P and Cohen E G D 2000 Microscopic chaos and diffusion *J. Stat. Phys.* **101** 775–817

[25] Friedrich R, Peinke J and Renner Ch 2000 How to quantify deterministic and random influences on the statistics of the foreign exchange market *Phys. Rev. Lett.* **84** 5224–7